Normalisation of the wave function

The schrödinger wave equation is linear and homogeneous in ψ and its derivatives (viz $\frac{\partial^2\psi}{\partial x^2}$ and $\frac{\partial\psi}{\partial t}$) That is why if any solution of it be multiplied by a constant then the resulting wave function will also be a solution. To avoid this arbitariness, it is necessary to impose normalisation condition.

Let us suppose $\psi_1(\vec{r},t)$ is a soln of the schrodinger eqn (1) and

$$\int_{-\infty}^{+\infty} |\psi_1(\vec{r},t)|^2 \, d^3r = N^2 \quad \text{---} (1)$$

where $|\psi_1(\vec{r},t)|^2 = \psi_1^*(\vec{r},t)\,\psi_1(\vec{r},t)$ is a +ve number and so also the integral. N^2 is therefore a real positive number and is called the norm of $\psi_1(\vec{r},t)$.

Now consider another wave function $\psi(\vec{r},t)$ which is related to $\psi_1(\vec{r},t)$ by the relation

$$\psi(\vec{r},t) = \frac{1}{N}\psi_1(\vec{r},t) \quad \text{---} (2)$$

Here $\psi(\vec{r},t)$ is a solution of eqn (1) since it differs from the solution $\psi_1(\vec{r},t)$ by the factor $\frac{1}{N}$ which is constant.

Using eqn (2) in (1) we get

$$\int_{-\infty}^{+\infty} N^2 |\psi(\vec{r},t)|^2 \, d^3r = N^2$$

or $$\int_{-\infty}^{+\infty} |\psi(\vec{r},t)|^2 \, d^3r = 1 \quad \text{---} (3)$$

The wave function $\psi(\vec{r},t)$ which satisfies

$-\frac{\hbar^2}{2m}(\nabla^2+V)\psi = E\psi$ ——— (1)

eqn (3) is said to be normalised. Hence a normalised wave function must have unit norm. Again N in eqn (2) must be finite since $|\psi(\vec{r},t)|^2$ is finite, then $|\vec{r}|$ is finite and $|\psi(\vec{r},t)|^2 \to 0$ as $|\vec{r}| \to \infty$. Eqn ③ is known as the condition of normalisation.

For one dimensional motion, the normalisation condition is given by

$$\int_{-\infty}^{+\infty} \psi^*(x,t)\,\psi(x,t)\,dx = \int_{-\infty}^{+\infty} |\psi(x,t)|^2\,dx = 1$$

Since for any solution of schrodinger eqn.

$$\int_{-\infty}^{+\infty} \psi^*(\vec{r},t)\,\psi(\vec{r},t)\,d^3r = \int_{-\infty}^{+\infty} |\psi(\vec{r},t)|^2\,d^3r$$

is independent of time

$$\therefore \int_{-\infty}^{+\infty} \psi^*(\vec{r},t)\,\psi(\vec{r},t)\,d^3r = \int_{-\infty}^{+\infty} \psi^*(\vec{r})\,\psi(\vec{r})\,d^3r$$

Hence the normalisation condition is

$$\int_{-\infty}^{+\infty} \psi^*(\vec{r})\,\psi(\vec{r})\,d^3r = 1 \quad \text{——④}$$

For one dimensional, normalisation condition is

$$\int_{-\infty}^{+\infty} \psi^*(x)\,\psi(x)\,dx = 1 \quad \text{——⑤}$$

Orthogonality of wave functions:

Consider a set of wave functions satisfying Schrodinger wave eqn, viz

$\psi_1(\vec{r}), \psi_2(\vec{r}), \psi_3(\vec{r}) \ldots$ etc.

such that any two functions $\psi_m(\vec{r})$ and $\psi_n(\vec{r})$ in the set satisfy the condition

$$\int \psi_m^*(\vec{r})\, \psi_n(\vec{r})\, d\tau = 0 \quad \text{---} \quad (6)$$

then the functions in the set are said to be orthogonal in analogy with the condition of orthogonality of two vectors.

If $\int \psi_m^*(\vec{r})\, \psi_m(\vec{r})\, d\tau = 1$ then each function is normalised. then the set of functions under consideration is called orthonormal set of wave functions.

The condition of orthogonality and normalization of the wave function can be expressed as

$$\int \psi_m^*(\vec{r})\, \psi_n(\vec{r})\, d\tau = \delta_{mn} \quad \text{---} \quad (7)$$

where δ_{mn} = Kronecker delta where

$$\delta_{mn} = \begin{cases} 1, & \text{when } m = n \\ 0, & \text{when } m \neq n \end{cases}$$

If the normalized orthogonal wave function are single valued, continuous, finite throughout the entire space of the system and satisfy the Schrodinger wave equation then they can be expanded as a series of other orthogonal wave functions of the same variable.

Show the wavefunction $\Psi(\vec{r},t) = \sum_n C_n \phi_n$ (or eqn 8) satisfying Schrödinger wave eqn

Show if $\psi(\vec{r})$ has the above properties and $\phi_1(\vec{r}), \phi_2(\vec{r}), \ldots \phi_n(\vec{r})$ are a set of mutually orthogonal wave functions then,

$$\psi(\vec{r}) = c_1\phi_1 + c_2\phi_2 + \ldots + c_m\phi_m \quad \text{— (8)}$$

where $c_1, c_2, \ldots c_m$ are constants.

Now multiplying eqn (8) by ϕ_m^* and then integrating over all space, we get

$$\int \phi_m^* \psi \, d\tau = c_1 \int \phi_m^* \phi_1 \, d\tau + c_2 \int \phi_m^* \phi_2 \, d\tau + \ldots + c_m \int \phi_m^* \phi_m \, d\tau \quad \text{— (9)}$$

Again since $\int \phi_m^* \phi_n \, d\tau = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$

So from eqn (9) we get

$$\int \phi_m^* \psi \, d\tau = c_m \quad \text{— (10)}$$

$\therefore$ the values of the expansion coefficients $c_1, c_2, \ldots c_m$ can be found out using eqn (10)

The complex conjugate of eqn (8) is

$$\psi^* = c_1^* \phi_1^* + c_2^* \phi_2^* + \ldots + c_m^* \phi_m^* + \ldots \quad \text{— (11)}$$

So $\psi^*\psi = [c_1^*\phi_1^* + c_2^*\phi_2^* + \ldots + c_m^*\phi_m^* + \ldots][c_1\phi_1 + c_2\phi_2 + \ldots + c_m\phi_m + \ldots]$

Again since ψ is normalized

$$\int \psi^*\psi \, d\tau = 1$$

$$\therefore \int (c_1^*\phi_1^* + c_2^*\phi_2^* + \ldots + c_m^*\phi_m^* + \ldots)(c_1\phi_1 + c_2\phi_2 + \ldots + c_m\phi_m + \ldots) d\tau = 1 \quad \text{— (12)}$$

or $c_1^*c_1 + c_2^*c_2 + \ldots + c_m^*c_m = 1$

$$\left[\because \int \phi_m \phi_n \, d\tau = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases}\right]$$

or $\sum_m c_m^* c_m = 1 \quad \text{— (13)}$

Stationary States:

If the probability density, $\rho = \psi^* \psi$ is independent of time for a particular state, then the state under consideration is called a stationary state.

Now for ~~various~~ the general solution of 3-D time dependent schrodinger eqn is given by

$$\psi(\vec{r},t) = \sum_n C_n \psi_n e^{-iE_n t/\hbar} \quad \text{—— (14)}$$

$$\psi^*(\vec{r},t) = \sum_m C_m^* \psi_m^* e^{iE_m t/\hbar}$$

the probability density

$$\rho(\vec{r},t) = \psi^*(\vec{r},t)\,\psi(\vec{r},t)$$

$$= \sum_n C_n^* C_n \psi_n^* \psi_n + \sum_m \sum_n C_m^* C_n \psi_m^* \psi_n e^{i(E_m - E_n)t/\hbar} \quad \text{—— (15)}$$

Eqn (15) shows that $\rho(\vec{r},t)$ depends on time t and the state expressed by eqn (14) is not a stationary state. $\rho(\vec{r},t)$ will be independent of t if C_n's are zero for all values except for one value E_k. In such case, the wave function will contain only first term and represented by

$$\psi(\vec{r},t) = C_k \psi_k e^{-iE_k t/\hbar} \quad \text{—— (16)}$$

Now for this energy state E_k

$$\psi^*(\vec{r},t)\,\psi(\vec{r},t) = C_k^* C_k \psi_k^* \psi_k = \rho(\vec{r}) \quad \text{—— (17)}$$

Thus $\rho(\vec{r})$ is independent of time and energy state represented by E_k is called stationary state. The solution represented by eq (16) is the stationary state solution of schrodinger eqn since ρ is independent of time.

Conservation of probability

According to the probability interpretation of the normalised wavefunction, the probability of finding the particle over all space is unity. This implies that the particle must be found somewhere in space. If the particle does not decay or disappear in some other way then above statement must be true at all values of time, in other words the total probability must be conserved i.e $|\psi(\vec{r},t)|^2$ must be independent of time.

$$\text{So } \frac{\partial}{\partial t}\int |\psi(\vec{r},t)|^2 d\tau = \int \frac{\partial}{\partial t}|\psi(\vec{r},t)|^2 d\tau$$

$$= \int \frac{\partial \rho(\vec{r},t)}{\partial t} d\tau = 0$$

where $\rho(\vec{r},t) = |\psi(\vec{r},t)|^2$

The time dependent 3-D Schrodinger eqn is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \quad \text{——— (18)}$$

Replacing ψ by its complex conjugate ψ^* we

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* \quad \text{——— (19)}$$

Multiplying (18) by ψ^* and (19) by ψ we get

$$i\hbar \psi^* \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi \quad \text{——— (20)}$$

$$-i\hbar \psi \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + V\psi\psi^* \quad \text{——— (21)}$$

Subtracting (21) from (20) we get

$$i\hbar\left(\psi^*\frac{\partial \psi}{\partial t} + \psi\frac{\partial \psi^*}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right)$$

$$\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} = -\frac{\hbar}{2im}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$$

$$\Rightarrow \frac{\partial}{\partial t}(\psi^*\psi) + \frac{i\hbar}{2m}\left[-\psi^* \vec{\nabla}\psi + \psi \vec{\nabla}\psi^*\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\psi^*\psi) + \frac{i\hbar}{2m}\left(\psi \vec{\nabla}\psi^* - \psi^* \vec{\nabla}\psi\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\psi^*\psi) + \frac{i\hbar}{2m}\left[\frac{\partial}{\partial x}\left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right)\right.$$

$$\left. + \frac{\partial}{\partial y}\left(\psi \frac{\partial \psi^*}{\partial y} - \psi^* \frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\psi \frac{\partial \psi^*}{\partial z} - \psi^* \frac{\partial \psi}{\partial z}\right)\right] = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{— (22)}$$

where $\rho(\vec{r},t) = \rho = \psi^*\psi$ — (23)

$$\vec{\nabla} \cdot \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \quad \text{— (24)}$$

with $J_x = \frac{i\hbar}{2m}\left[\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right]$ — 25

$$J_y = \frac{i\hbar}{2m}\left[\psi \frac{\partial \psi^*}{\partial y} - \psi^* \frac{\partial \psi}{\partial y}\right] \quad \text{— (26)}$$

$$J_z = \frac{i\hbar}{2m}\left[\psi \frac{\partial \psi^*}{\partial z} - \psi^* \frac{\partial \psi}{\partial z}\right] \quad \text{— (27)}$$

$$\therefore \vec{J} = \vec{J}(\vec{r},t) = \frac{i\hbar}{2m}\left[\psi \nabla \psi^* - \psi^* \nabla \psi\right]$$

— (28)

Now differentiating $\int \rho(\vec{r},t)\, d\tau$ over all space with respect to time

$$\frac{\partial}{\partial t}\int_\tau \rho(\vec{r},t)\, d\tau = \int_\tau \frac{\partial}{\partial t}(\psi^*\psi)\, d\tau$$

$$= -\int_\tau \vec{\nabla} \cdot \vec{J}\, d\tau$$

$$\Rightarrow \frac{\partial}{\partial t}\int_\tau \rho(\vec{r},t)\, d\tau = -\oint_S \vec{J} \cdot \hat{n}\, ds \quad \text{— (29)}$$

By Gauss divergence theorem

where S is closed surface enclosing volume τ.

Since the volume integral is over all space & the surface S is at ∞ i.e $|\vec{r}| \to \infty$. This implies ψ and $\nabla\psi$ vanish at the surface S and $\vec{J}$ also vanish as $|\vec{r}| \to \infty$.

So $$\oint_S \vec{J} \cdot \hat{n}\, ds = 0$$

Hence $$\frac{\partial}{\partial t} \int_\Omega \rho(\vec{r},t)\, d\Omega = 0$$

$$\Rightarrow \int_\Omega \rho(\vec{r},t)\, d\Omega \text{ is independent of time}$$

and satisfies the conservation of total probability.

Probability Current density:

Eqn (22) is given by

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \vec{\nabla} \cdot \vec{J}(\vec{r},t) = 0$$ This is the familiar equation of continuity in hydrodynamics where $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ represent density of matter and current density. In the present case $\rho = \psi^*\psi$ represents probability density and $\vec{J}$ is the probability current density given by eqn (28)

A state for which $\vec{\nabla} \cdot \vec{J} = 0$, $\rho(\vec{r},t) = \text{const}$ in time then the corresponding state is a stationary state.

Conditions of Admissibility of the wave function

The 3-D time dependent Schrodinger wave equation for a particle is given by

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This eqn being a second order linear differential equation may give rise to several solutions. But every solution of the equation cannot represent a matter wave associated with the particle. Only those solutions satisfying certain conditions may represent a particle. For those wave functions $|\psi(\vec{r},t)|^2 d\tau$ should represent the probability of finding the particle in the volume element $d\tau$.

The admissibility conditions are:

(1) $\psi(\vec{r},t)$ must be finite everywhere in space. This is a must because $|\psi(\vec{r},t)|^2 d\tau$ for volume element $d\tau$ should be between 0 and 1.

(2) $\psi(\vec{r},t)$ must be single valued everywhere.

(3) ψ and its first order derivatives with respect to its variables must be continuous.

(4) ψ must be square integrable so that the integral of its modulus square is finite

$$I = \int_a |\psi|^2 d\tau \text{ is finite or zero.}$$

(5) ψ is zero at any surface at which there is an infinite potential step; the component of $\nabla\psi$ normal to the surface is undetermined.

A wave function which satisfies all the above conditions is an admissible wave function or well behaved wave function.

(4) $\psi(x)$ must be square integrable so that the integral of its modulus squared is finite.

$$P = \int_{\tau} |\psi|^2 \, d\tau$$ is finite or zero.

(5) ψ is zero at any surface at which there is an infinite potential step; one component of $\nabla\psi$ normal to the surface is undetermined.

A wavefunction which satisfies all the above conditions is an admissible wave function or acceptable wave function.

Problems

(1) The wave function of a particle confined to a box of length L is

$$\psi(x) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{\pi x}{L}$$ in the region $0 < x < L$ and zero everywhere else. Calculate the probability of finding the particle in the region $0 < x < L/2$.

Soln The probability of finding the particle in interval δx at distance x is

$$P(x)\,\delta x = \psi^*(x)\,\psi(x)\,\delta x$$

Here $\psi(x) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{\pi x}{L} = \psi^*(x)$

$$\therefore P(x)\,\delta x = \frac{2}{L}\sin^2\frac{\pi x}{L}\,\delta x$$

Total probability in region $0 < x < L/2$ is

$$P = \int_0^{L/2} P(x)\,dx = \frac{2}{L}\int_0^{L/2} \sin^2\frac{\pi x}{L}\,dx = \frac{2}{L}\cdot\frac{1}{2}\int_0^{L/2}\left(1 - \cos\frac{2\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[\frac{L}{2} - \sin\frac{2\pi x}{L}\Big|_0^{L/2}\right] = \frac{1}{L}\left[\frac{L}{2} - 0\right] = \frac{1}{2}$$ Ans

$1 - 2\sin^2\theta = \cos 2\theta$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

h Schrödinger..

2) Normalised the one dimensional wave function given by $\psi(x) = A \sin \frac{\pi x}{l}$, $0 < x < l$; $= 0$ outside

Sch The wave function $\psi(x)$ is said to be normalised if it satisfies $\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)\,dx = 1$

As the given wave function $\psi(x)$ exists only in the region $0 < x < l$ and is zero outside so

$$\int_0^l \psi^*(x)\psi(x)\,dx = 1$$

$$\Rightarrow \int_0^l A^* \left(\sin \frac{\pi x}{l}\right) A \sin \frac{\pi x}{l}\,dx = 1$$

$$\Rightarrow A^*A \int_0^l \sin^2 \frac{\pi x}{l}\,dx = 1$$

$$\Rightarrow A^*A \left[\int_0^l \frac{\left(1 - \cos \frac{2\pi x}{l}\right)}{2} dx\right] = 1$$

$$\Rightarrow A^*A \left(\int_0^l dx - \int_0^l \cos \frac{2\pi x}{l} dx\right) = 2$$

$$\Rightarrow A^*A \left[l - \frac{1}{2\pi/l} \sin \frac{2\pi}{l} \cdot x \Big|_0^l\right] = 2$$

$$\Rightarrow A^*A \left[l - \frac{l}{2\pi} (\sin 2\pi - \sin 0)\right] = 2$$

$$\Rightarrow A^*A \left[l - \frac{l}{2\pi}\right] = 2$$

$$\Rightarrow |A|^2 = \frac{2}{l} \Rightarrow A = \sqrt{\frac{2}{l}}$$

Assuming A to be real

Hence normalised wave function is

$$\psi(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{\pi x}{l}\right)$$

(2) The wave function of a particle confined to a box of length L is $\psi(x) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{\pi x}{L}$ in the region $0 < x < L$ and zero everywhere else.

Calculate the probability of finding the particle in the region $0 < x < L/2$.

Soln The probability of finding the particle in interval δx at a distance x is

$$P(x)\,\delta x = \psi^*(x)\,\psi(x)\,\delta x$$

Here $\psi(x) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{\pi x}{L}$

$\psi^*(x) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\frac{\pi x}{L}$ (real)

$$P(x)\,\delta x = \frac{2}{L}\sin^2\frac{\pi x}{L}\,\delta x$$

The probability of finding particle in the region $0 < x < L/2$ is

$$P = \int_0^{L/2} P(x)\,dx = \frac{2}{L}\int_0^{L/2} \sin^2\frac{\pi x}{L}\,dx$$

$$= \frac{2}{L}\cdot\frac{1}{2}\int_0^{L/2}\left(1-\cos 2\frac{\pi x}{L}\right)dx$$

$$= \frac{1}{L}\cdot\left[\frac{L}{2} - 0\right] = \frac{1}{2}$$

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(3) The wave function of a particle is given by $\psi(x) = Ce^{-\alpha x^2}$, $-\infty < x < +\infty$ where C and α are constants. Calculate the probability of finding the particle in the region $0 < x < \infty$.

wave equation is linear.

Sol^n

The probability of finding the particle in the region $0<x<a$ is given by

$$P = \int_0^a \psi^*(x)\,\psi(x)\,dx = \int_0^a C^* e^{-\alpha x^2} \cdot C e^{-\alpha x^2} dx$$

$$= C^* C \int_0^a e^{-2\alpha x^2} dx = C^2 \cdot \frac{1}{2}\sqrt{\frac{\pi}{2\alpha}}$$

(S.I: $\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$) (1)

Now to normalize the wave function

$$\int_{-a}^{+a} |\psi(x)|^2 dx = 1$$

$$\Rightarrow \int_{-a}^{+a} |C|^2 e^{-2\alpha x^2} dx = 1$$

$$\Rightarrow |C|^2 \cdot 2 \cdot \frac{1}{2}\sqrt{\frac{\pi}{2\alpha}} = 1 \Rightarrow |C|^2 = \sqrt{\frac{2\alpha^2}{\pi}}$$

Answer (c)

$$\therefore P = \sqrt{\frac{2\alpha^2}{\pi}} \cdot \frac{1}{2}\sqrt{\frac{\pi}{2\alpha}} = \frac{1}{2}$$

Hence probability of finding the particle is $\frac{1}{2}$.

(4) Calculate the normalisation constant for a wave function (at $t=0$) given by $\psi(x) = a e^{-\frac{\alpha x^2}{2}} e^{ikx}$. Determine probability density and probability current density.

According to the condition of normalisation

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = |a|^2 \int_{-\infty}^{+\infty} e^{-\alpha^2 x^2} dx = 1$$

$$\Rightarrow |a|^2 2 \cdot \frac{\sqrt{\pi}}{\alpha^2} = 1$$

$$\Rightarrow |a| = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}$$

= normalisation constant of wavefunction

probability density $= P(x) = \psi^* a \psi a = |a|^2 \cdot e^{-\frac{2\alpha^2 x^2}{2}} \cdot 1 = |a|^2 e^{-\alpha^2 x^2}$

$$= \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2} \quad \text{---}$$

$$\frac{\partial \psi}{\partial x} = a e^{-\frac{\alpha^2 x^2}{2} + ikx} \cdot a(ik - \alpha^2 x)$$

$$= a(ik - \alpha^2 x) e^{-\alpha^2 x^2/2} e^{ikx}$$

$$\frac{\partial \psi^*}{\partial x} = a^*(-ik - \alpha^2 x) e^{-\alpha^2 x^2/2} e^{-ikx}$$

The probability current density $\hat{i}$

$$J_x = \frac{i\hbar}{2m}\left[\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right]$$

$$= \frac{i\hbar}{2m} |a|^2 \left[-ik - \alpha^2 x - ik + \alpha^2 x\right) e^{-\alpha^2 x^2}$$

$$= \frac{\hbar k}{m} |a|^2 e^{-\alpha^2 x^2} = \frac{\hbar k}{m} P$$

A plane wave is given by the wave function $\psi(x) = Ae^{ikx}$ in one dimension. find the probability current density.

Sol: The probability current density is given by

$$J = \frac{\hbar}{2mi}\left[\psi^* \nabla\psi - \psi \nabla\psi^*\right]$$

In one dimension

$$J = \frac{\hbar}{2im}\left[\psi^* \frac{\partial\psi}{\partial x} - \psi \frac{\partial\psi^*}{\partial x}\right]$$

$\psi = Ae^{ikx}$, $\psi^* = A^* e^{-ikx}$

$\frac{\partial\psi}{\partial x} = ikAe^{ikx}$, $\frac{\partial\psi^*}{\partial x} = -ikA^* e^{-ikx}$

$$J = \frac{\hbar}{2im}\left[A^* e^{-ikx} \cdot ikAe^{ikx} - ikAe^{ikx}(-ik)A^* e^{-ikx}\right]$$

$$= \frac{\hbar}{2im}\left[ik|A|^2 + ik|A|^2\right] = \frac{\hbar}{2im} 2ik|A|^2$$

$$= \frac{\hbar k}{m}|A|^2 = v|A|^2$$

where v = velocity = $\frac{\hbar k}{m}$

$$\left(\frac{\hbar k}{m} = \frac{h}{2\pi}\cdot\frac{2\pi}{\lambda}\cdot\frac{1}{m} = \frac{h}{h/mv}\cdot\frac{1}{m} = v\right)$$

$\therefore J = v|A|^2$

The probability density of the particle is

$$\rho = \psi^*\psi = A^* e^{-ikx} \cdot Ae^{ikx} = |A|^2$$

$\therefore$ The probability current density is

J = velocity $\times$ probability density

$= v\rho$ = flow of probability per unit area.

(b) If $\psi_1(x,t)$ and $\psi_2(x,t)$ are both the solutions of Schrodinger's wave equation for a given potential $V(x,t)$, then show that $\psi = a_1\psi_1 + a_2\psi_2$ in which a_1 and a_2 are arbitrary constants is also a solution.

Sol: One dimensional time dependent Schrodinger equation is $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t}$ ——(1)

Since $\psi_1(x,t)$ and $\psi_2(x,t)$ are both solutions of the above equation we have

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial x^2} + V\psi_1 = i\hbar\frac{\partial\psi_1}{\partial t} \quad ——(2)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} + V\psi_2 = i\hbar\frac{\partial\psi_2}{\partial t} \quad ——(3)$$

Multiplying 2 by a_1 and 3 by a_2 and then adding we get

$$-\frac{\hbar^2}{2m}\frac{\partial^2 a_1\psi_1}{\partial x^2} + V a_1\psi_1 = i\hbar\frac{\partial a_1\psi_1}{\partial t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 a_2\psi_2}{\partial x^2} + V a_2\psi_2 = i\hbar\frac{\partial a_2\psi_2}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2(a_1\psi_1 + a_2\psi_2)}{\partial x^2} + V(a_1\psi_1 + a_2\psi_2) = i\hbar\frac{\partial}{\partial t}(a_1\psi_1 + a_2\psi_2) \quad ——(4)$$

Comparing (1) and (4)

$$\psi = a_1\psi_1 + a_2\psi_2 \quad ——(5)$$

Thus eqn (5) is also a solution of time dependent wave equation.

Q A one dimensional wave function is given by $\psi(x) = \sqrt{a}\, e^{-ax}$, find the probability of finding the particle between $x = \frac{1}{a}$ and $x = 2/a$

Sol Probability of finding the particle between x and $x+dx$ is

$$P(x)\,dx = \psi^* \psi\, dx = \sqrt{a}\, e^{-ax} \cdot \sqrt{a}\, e^{-ax}\, dx$$

$$= a\, e^{-2ax}\, dx$$

Probability of finding the particle between $x = \frac{1}{a}$ and $x = \frac{2}{a}$ is

$$P = \int_{1/a}^{2/a} a e^{-2ax}\, dx = a \left| \frac{e^{-2ax}}{-2a} \right|_{1/a}^{2/a}$$

$$= -\frac{1}{2}\left[e^{-2a \cdot \frac{2}{a}} - e^{-2a \cdot \frac{1}{a}} \right]$$

$$= -\frac{1}{2}\left[e^{-4} - e^{-2} \right] = \frac{e^2 - e^4}{2}$$ Ans

Q The wave function of a particle constrained to move along x ($-\infty < x < \infty$) at a certain instant is given by $\psi(x) = A e^{-x^2/a^2 + ibx}$ a, b are real constants. find the normalisation constant (A),

The condition of normalisation of the wavefunction of $\psi(x)$ is $\int_{-\infty}^{+\infty} \psi^*(x)\, \psi(x)\, dx = 1$

But $\psi(x) = A e^{-\frac{x^2}{a^2} + ibx}$, $\psi^*(x) = A^* e^{-\frac{x^2}{a^2} - ibx}$

$$\int_{-\infty}^{+\infty} |A|^2 e^{-2x^2/a^2}\, dx = 1 \quad \text{---(1)}$$

Putting $\frac{2x^2}{a^2} = z$, $x^2 = \frac{a^2 z}{2} \Rightarrow x = a\sqrt{\frac{z}{2}}$

$$\Rightarrow 2x\,dx = \frac{a^2}{2}\,dz$$

$$\Rightarrow dx = \frac{a^2}{4x}\,dz$$

$$= \frac{a^2\,dz}{4\,a\,\frac{z^{1/2}}{a^{1/2}}}$$

$$= \frac{a\,z^{-\frac{1}{2}}\,dz}{2\sqrt{2}}$$

Eqn (1) becomes

$$A^2 \int_{-\infty}^{+\infty} e^{-z} \frac{a\,z^{-\frac{1}{2}}}{2\sqrt{2}}\,dz = A^2 \frac{a}{2\sqrt{2}} \int_{-\infty}^{+\infty} e^{-z} z^{-\frac{1}{2}}\,dz$$

$$= \frac{A^2 a}{2\sqrt{2}} \int_{-\infty}^{+\infty} e^{-z} z^{\frac{1}{2}-1}\,dz = A^2 \frac{a}{2\sqrt{2}} \times 2\,\Gamma\left(\frac{1}{2}\right)$$

$$= A^2 \frac{2a}{2\sqrt{2}} \sqrt{\pi}$$

$$\Rightarrow A^2 \frac{a}{\sqrt{2}} \sqrt{\pi} = 1 \Rightarrow A^2 = \frac{2\sqrt{2}}{a\sqrt{\pi}} = \frac{2}{a}\sqrt{\frac{2}{\pi}}$$

$$\Rightarrow A = \left(\frac{2}{a}\sqrt{\frac{2}{\pi}}\right)^{1/2}$$